

On the asymptotics of Kronecker coefficients

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1 Introduction

Kronecker coefficients are the structure constants for the tensor products of irreducible representations of symmetric groups. Their computation is thus an old and important problem in finite group theory. Through Schur-Weyl duality, they can be interpreted as multiplicities in Schur powers of tensor products and their computation becomes a problem in invariant theory. Applying the Borel-Weil theorem we can see Schur powers of tensor products as spaces of sections of equivariant line bundles on flag varieties, and the problem can be interpreted as a question about Hamiltonian actions on symplectic manifolds. It can also be considered in relation with marginal problems in quantum information theory. Each of these perspectives gives access to interesting pieces of information about Kronecker coefficients, which can be very hard to obtain, or even to guess, from a different perspective. But although we have these several points of views on the Kronecker coefficients, each revealing some part of their properties, they remain quite mysterious, and very basic problems seem completely out of reach at this moment. Let us mention a few important ones.

1. No combinatorial formula is known. We would like the Kronecker coefficients to count some combinatorial objects. Or points in polytopes.
2. A few linear conditions for Kronecker coefficients to be non zero are known. We would like to know all of them. In the terminology of this paper, we would like to know the faces of the Kronecker polyhedra.
3. These polyhedra being described, we would like to know how far we are from being able to decide whether a Kronecker coefficient is zero or not. This is a saturation type problem, related to Mulmuley's conjecture that this decision problem is in P.
4. We would like to understand stretched Kronecker coefficients. They are given by certain quasipolynomials, which we would like to be able to compute efficiently.

The analogue problems for Littlewood-Richardson coefficients have been solved. The Littlewood-Richardson rule gives a combinatorial recipe for their computation, which can be interpreted as a count of integral points in polytopes, once translated in terms of hives or honeycombs, for example. The full list of linear inequalities is known and gives a solution to the famous Horn problem (see eg [7] for a survey). They are enough to decide whether a Littlewood-Richardson coefficients is zero or not, a decision problem which is definitely in P. Finally the stretched versions have been studied and are in fact given by polynomials, not just quasipolynomials [22].

This gives some hope for the seemingly similar Kronecker coefficients. There are two different ways to appreciate the connections between these two families of numbers. Littlewood-Richardson coefficients are multiplicities in Schur powers of direct sums rather than tensor products. They are also special Kronecker coefficients: in fact, quite remarkably there is a special facet of the Kronecker polyhedron which is exactly a Littlewood-Richardson polyhedron. The latter is known to be generated by the triple of partitions for which the Littlewood-Richardson coefficient is exactly one, and there is a remarkable equivalence

$$c_{\lambda,\mu}^{\nu} = 1 \iff c_{k\lambda,k\mu}^{k\nu} = 1 \quad \forall k \geq 1.$$

Such a property will certainly not hold for Kronecker coefficients, but it suggests to introduce the following definition:

Definition. A triple of partitions (α, β, γ) is *weakly stable* if the Kronecker coefficients

$$g(k\alpha, k\beta, k\gamma) = 1 \quad \forall k \geq 1.$$

It is *stable* if $g(\alpha, \beta, \gamma) \neq 0$ and for any triple (λ, μ, ν) , the sequence of Kronecker coefficients $g(\lambda + k\alpha, \mu + k\beta, \nu + k\gamma)$ is bounded (or equivalently, eventually constant).

Stability was introduced in [24] by Stembridge, who proved that it implies weak stability and conjectured that the two notions are in fact equivalent. From the Kronecker polyhedron perspective, stable or weakly stable triples will correspond to very special boundary points.

One of the main objectives of the present paper is to explain how to construct large families of stable triples. Moreover we will be able to describe the Kronecker polyhedron locally around these special points, including by giving explicit equations of the facets they belong to.

Our main tool for studying the Kronecker coefficients will be Taylor expansion. This might look strange at first sight, but recall that using Schur-Weyl duality and the Borel-Weil theorem we can interpret them in terms of spaces of sections of line bundles on flag manifolds. In this context it is very natural to use some basic analytic tools, like Taylor expansion, in order to analyze such sections. We will use these Taylor expansions around certain flag subvarieties, and the fact that these subvarieties allow us to control the Kronecker coefficients will depend on certain combinatorial properties of standard tableaux. The key concept here is that of additivity, studied in detail by Vallejo [25, 27]. In fact this concept already appears in [12], where very similar ideas are introduced and used to understand certain asymptotic properties of plethysms. Section 3 of [12] is in fact already devoted to Kronecker coefficients: we explained how the method could be used in this context, what was the role of the additivity property, how we could deduce infinite families of stable triples, and much more. In the present paper we explain our approach in greater detail in the specific context of Kronecker coefficients, and go a bit further than in [12]. Our results are the following:

1. We show that in a very general setting, stable triples can be obtained through equivariant embeddings of flag manifolds. The asymptotics of the Kronecker coefficients are then governed by the properties of the normal bundle of the embedding, whose weights must be contained in an open half space. When this occurs, a very general stability phenomenon can be observed, and the limit multiplicities can be computed. In particular the local structure of the Kronecker polyhedron can be described (Theorem 2).
2. The simplest example is a Segre embedding of a product of projective spaces. This takes care of the very first instance of stability, discovered by Murnaghan a very long time ago [19, 20], which corresponds to the simplest stable triple $(1, 1, 1)$. This immediately leads to an expression of the limit multiplicities, traditionally called *reduced Kronecker coefficients*. Moreover this can readily be generalized: considering the product of a projective space

by a Grassmannian we recover a stability property called *k-stability* by Pak and Panova [21], and we are able to compute the limit multiplicities (Proposition 3). Of course this further generalizes to a general product of Grassmannians, which yields a new stability property. In fact, what we show is that $(1^{ab}, a^b, b^a)$ is always a stable triple (Proposition 4). Moreover we can in principle compute the limit multiplicities.

3. Turning to products of flag varieties we explain why the convexity property of the normal bundle of an embedding defined by a standard tableau is in fact equivalent to the property that the tableau is additive (Proposition 6). In fact this is already contained in [12], even in its version generalized to multitableaux. What we explain in the present paper is how the additive tableaux define minimal regular faces of the Kronecker polyhedron (Proposition 9). Moreover, we describe all the facets containing these minimal faces in terms of special tableaux that we call maximal relaxations (Proposition 10). The corresponding inequalities are completely explicit (Propositions 11 and 12).
4. Each of these special facets consists in stable triples, that we therefore obtain in abundance. Moreover, for each of these triples there is a corresponding notion of reduced Kronecker coefficients. When the stable triple is regular, which is the generic case, we compute this reduced Kronecker coefficient as a number of integral points in an explicit polytope (Proposition 8). As we mentioned above, this is something we would very much like to be able to do for general Kronecker coefficients.
5. The last section of the paper is devoted to certain Kronecker coefficients for partitions of rectangular shape. There are some nice connections with the beautiful theory of θ -groups of Vinberg and Kac (Proposition 13). In particular we prove an identity suggested by Stembridge in [24] by relating it to the affine Dynkin diagram of type E_6 (see Proposition 14, which also contains an interesting identity coming from affine D_4).

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2 From Kronecker to Borel-Weil

2.1 Schur-Weyl duality

For any integer n , the irreducible complex representations of the symmetric group S_n are naturally parametrized by partitions of n [13]. We denote by $[\lambda]$ the representation defined by the partition λ of n (we use the notation $\lambda \vdash n$ to express that λ is a partition of n , in which case we also say that λ has size n). The Kronecker coefficients can be defined as dimensions of spaces of S_n -invariants inside triple tensor products:

$$g(\lambda, \mu, \nu) = \dim([\lambda] \otimes [\mu] \otimes [\nu])^{S_n}.$$

Since the irreducible representations of S_n are defined over the reals, they are self-dual and therefore, we can also interpret the Kronecker coefficients as multiplicities in tensor products:

$$[\lambda] \otimes [\mu] = \bigoplus_{\nu \vdash n} g(\lambda, \mu, \nu) [\nu].$$

Schur-Weyl duality allows to switch from representations of symmetric groups to representations of general linear groups. Recall that irreducible polynomial representations of $GL(V)$ are

parametrized by partitions with at most $d = \dim V$ non zero parts. We denote by $S_\lambda V$ the representation defined by the partition λ . Schur-Weyl duality can be stated as an isomorphism

$$V^{\otimes n} = \bigoplus_{\lambda \vdash n} [\lambda] \otimes S_\lambda V$$

between $S_n \times GL(V)$ -modules. Here the summation is over all partitions λ of n with at most d non zero parts (the number of non zero parts will be called the length and denoted $\ell(\lambda)$). A straightforward consequence is that

$$g(\lambda, \mu, \nu) = \text{mult}(S_\lambda V \otimes S_\mu W, S_\nu(V \otimes W)),$$

as soon as the dimensions of V and W are large enough; more precisely, as soon as $\dim(V) \geq \ell(\lambda)$ and $\dim(W) \geq \ell(\mu)$. Note that consequently, we get the classical result that

$$g(\lambda, \mu, \nu) = 0 \quad \text{if } \ell(\nu) > \ell(\lambda) + \ell(\mu).$$

2.2 The Borel-Weil theorem and its consequences

The $GL(U)$ -representation $S_\lambda U$ is called a Schur module. It is defined more generally for λ an arbitrary non increasing sequence of integers of length equal to the dimension of U . The Borel-Weil theorem asserts that such a Schur module can be realized as a space of sections of a linearized line bundle L_λ on the complete flag variety $Fl(U)$. We can therefore understand the Kronecker coefficients as the multiplicities in the decomposition of

$$S_\nu(V \otimes W) = H^0(Fl(V \otimes W), L_\nu)$$

into irreducible $GL(V) \times GL(W)$ -modules.

A direct (well-known) consequence of the Borel-Weil theorem is that the direct sum of all the Schur powers of a given vector space U has a natural algebra structure. This algebra is finitely generated, and therefore if we let $U = V \otimes W$, the subalgebra of $GL(V) \times GL(W)$ -covariants is also finitely generated (this is a consequence of the fact that the unipotent radical of a reductive algebraic group is a Grosshans subgroup). This implies the following result. Consider

$$Kron_{a,b,c} := \{(\lambda, \mu, \nu), \ell(\lambda) \leq a, \ell(\mu) \leq b, \ell(\nu) \leq c, g(\lambda, \mu, \nu) \neq 0\}.$$

Proposition 1 (Semigroup property) *$Kron_{a,b,c}$ is a finitely generated semigroup.*

(The semigroup property is Conjecture 7.1.4 in [10], where finite generation is also a conjecture.) Moreover, since the covariant algebra is described in terms of spaces of sections of line bundles, it has no zero divisor. This implies the following monotonicity property:

Proposition 2 (Monotonicity) *If $g(\lambda, \mu, \nu) \neq 0$, then for any triple (α, β, γ) ,*

$$g(\alpha + \lambda, \beta + \mu, \gamma + \nu) \geq g(\alpha, \beta, \gamma).$$

Remark. In fact a stronger property is true: there exists a natural map

$$([\lambda] \otimes [\mu] \otimes [\nu])^{S_n} \otimes ([\alpha] \otimes [\beta] \otimes [\gamma])^{S_p} \longrightarrow ([\lambda + \alpha] \otimes [\mu + \beta] \otimes [\nu + \gamma])^{S_{n+p}}$$

which is non zero on decomposable tensors. It could be interesting to understand this map better.

2.3 The Kronecker polyhedron

The semigroup $Kron_{a,b,c}$ lives inside \mathbf{Z}^{a+b+c} , more precisely inside a codimension two sublattice because of the obvious condition $|\lambda| = |\mu| = |\nu|$ for a Kronecker coefficient $g(\lambda, \mu, \nu)$ to be non zero. Consider the cone generated by $Kron_{a,b,c}$. The finite generation of the latter implies that this is a rational polyhedral cone $PKron_{a,b,c}$, defined by some finite list of linear inequalities.

The interpretation in terms of sections of line bundles on flag manifolds allows to understand this polytope as a moment polytope, and to use the powerful results that has been obtained in this context using the tools of Geometric Invariant Theory, or in an even broader, not necessarily algebraic context, those of Symplectic Geometry.

On the GIT side, there exist general statements that allow, in principle, to determine moment polytopes and in particular the Kronecker polyhedron $PKron_{a,b,c}$. For example Klyachko in [10] suggested to use the results of Berenstein and Sjamaar, and applied them in small cases. This method has recently been improved in [29]. This approach has two important limitations. First, it describes the moment polytope by a collection of inequalities which is in general far from being minimal: this means that even if one was able to list these inequalities, many of them would in fact not correspond to any facet of the polytope and would be redundant with the other ones. Second, even if far from being satisfactory, the effective production of this collection of inequalities would require the computation of certain Schubert constants in some complicated homogeneous spaces, which seems combinatorially an extremely hard challenge. More precisely, one would need to decide whether certain Schubert constants are zero or not, which is certainly more accessible than computing them but is certainly far beyond our current level of understanding.

The first issue has been essentially solved by Ressayre [23], who devised a way to produce, in principle, a minimal list of inequalities of certain moment polytopes: that is, a list of its facets, or codimension one faces. This approach has been remarkably successful for Littlewood-Richardson coefficients. However, applying them concretely to the Kronecker polyhedron $PKron_{a,b,c}$ seems completely untractable at the moment.

At least do we know the Kronecker polyhedron $PKron_{a,b,c}$ for small values of a, b, c . For example $([10, 6])$ $PKron_{3,3,3}$ is defined by the following seven inequalities, and those obtained by permuting the partitions λ, μ, ν :

$$\begin{aligned} \lambda_1 + \lambda_2 &\leq \mu_1 + \mu_2 + \nu_1 + \nu_2, \\ \lambda_1 + \lambda_3 &\leq \mu_1 + \mu_2 + \nu_1 + \nu_3, \\ \lambda_2 + \lambda_3 &\leq \mu_1 + \mu_2 + \nu_2 + \nu_3, \\ \lambda_1 + 2\lambda_2 &\leq \mu_1 + 2\mu_2 + \nu_1 + 2\nu_2, \\ \lambda_2 + 2\lambda_1 &\leq \mu_1 + 2\mu_2 + \nu_2 + 2\nu_1, \\ \lambda_3 + 2\lambda_2 &\leq \mu_1 + 2\mu_2 + \nu_3 + 2\nu_2, \\ \lambda_3 + 2\lambda_1 &\leq \mu_2 + 2\mu_1 + \nu_2 + 2\nu_3. \end{aligned}$$

Klyachko made extensive computations showing an overwhelming growth of complexity when the parameters increase. For example he claims that $PKron_{2,3,6}$ is defined by 41 inequalities, $PKron_{2,4,8}$ by 234 inequalities and $PKron_{3,3,9}$ by no less than 387 inequalities!

2.4 The quasipolynomiality property

The problem of understanding multiplicities in spaces of sections of line bundles is at the core of the study of Hamiltonian actions on symplectic manifolds. The *Quantization commutes with Reduction* type results have very strong consequences in the context we are interested in. In the algebraic setting, one starts with a smooth projective complex variety M with an action of a reductive group G , and an ample G -linearized line bundle L on M . In this context one considers

the virtual G -module

$$RR(M, L) = \oplus_{q \geq 0} (-1)^q H^q(M, L),$$

whose dimension is given by the Grothendieck-Riemann-Roch formula. It is then a very general statement, due to Meinrenken and Sjamaar [17] (see also [28]), that for any dominant weight α , the multiplicity of the irreducible G -module of highest weight $k\alpha$ inside the virtual G -module $RR(M, L^k)$ is given by a quasipolynomial function of k .

Of course we will apply this result to $G = GL(U) \times GL(V) \times GL(W)$ acting on $M = \mathbf{P}(U \otimes V \otimes W)$, and L the hyperplane line bundle on this projective space. Since ample line bundles on projective spaces have no higher cohomology, we are in the favorable situation where $RR(M, L^k)$ is just the actual G -module $H^0(M, L^k) = S^k(U \otimes V \otimes W)^*$, whose multiplicities are precisely the Kronecker coefficients. Applying the full force of Meinrenken and Sjamaar's results, we deduce the following statement:

Theorem 1 *The stretched Kronecker coefficient $g(k\lambda, k\mu, k\nu)$ is a piecewise quasi-polynomial function of (k, λ, μ, ν) . More precisely, there is a finite decomposition of the Kronecker polyhedron $PKron_{a,b,c}$ into closed polyhedral subcones called chambers, and for each chamber C a quasi-polynomial $p_C(k, \lambda, \mu, \nu)$, such that*

$$g(k\lambda, k\mu, k\nu) = p_C(k, \lambda, \mu, \nu)$$

whenever (λ, μ, ν) belongs to C .

Corollary 1 *For any triple (λ, μ, ν) , the stretched Kronecker coefficient $g(k\lambda, k\mu, k\nu)$ is a quasi-polynomial function of $k \geq 0$.*

The monotonicity property of Kronecker coefficients easily implies that whenever (λ, μ, ν) is in the interior of the Kronecker polyhedron, the stretched Kronecker coefficient $g(k\lambda, k\mu, k\nu)$ grows as fast as possible, in the sense that if $g(\lambda, \mu, \nu) \neq 0$, then

$$g(k\lambda, k\mu, k\nu) \simeq C(\lambda, \mu, \nu) k^{n_{gen}}.$$

Here we denoted by n_{gen} the generic order of growth, which is also the generic dimension of the GIT-quotient $M//G$. For $PKron_{a,b,c}$ we get the value

$$n_{gen} = abc - a^2 - b^2 - c^2 + 2.$$

Moreover the coefficient $C(\lambda, \mu, \nu)$ can be expressed as the volume of the so-called reduced space $M_{(\lambda, \mu, \nu)}$ (see [28], page 21). This volume function is given by the Duistermaat-Heckman measure, which is piecewise polynomial, and not just quasipolynomial.

Conversely, if $g(\lambda, \mu, \nu) \neq 0$ and $g(k\lambda, k\mu, k\nu)$ grows like k^n for some $n < n_{gen}$, then the triple (λ, μ, ν) must belong to the boundary of $PKron_{a,b,c}$. The extreme case is when $n = 0$. As observed in [24], this can happen only if $g(k\lambda, k\mu, k\nu) = 1$ for all $k \geq 1$. Otherwise said, (λ, μ, ν) is a *weakly stable triple*.

The quasi-polynomiality property has attracted the attention of several authors. Most of them realized that general arguments based on finite generation of covariant rings implied the eventual quasipolynomiality of the stretched Kronecker coefficients, that is, $g(k\lambda, k\mu, k\nu)$ is given by some quasipolynomial for large enough k . The much stronger property that the stretched Kronecker coefficients are quasipolynomial right from the beginning seem to be much less accessible using algebraic methods only. It was asked as a question in [24]. It is also discussed in [18] and some explicit computations appear in [1].

Remark 1. One interesting implication of the quasipolynomiality property is that, knowing the Kronecker coefficients asymptotically, in fact we know them completely. Could this be useful to find triples (λ, μ, ν) in the Kronecker polyhedron, such that $g(\lambda, \mu, \nu) = 0$? The study of such *holes* seems extremely challenging, but is of the greatest importance for Geometric Complexity Theory [4].

Remark 2. We have noticed that although the stretched Kronecker coefficient $g(k\lambda, k\mu, k\nu)$ is a quasi-polynomial function of $k \geq 0$, its highest order term is really polynomial. More generally, examples show that the period of the coefficients seem to increase when one consider terms of lower degrees. It would be interesting to prove an explicit statement of this type.

3 Kronecker coefficients and Taylor expansions

3.1 The general setup

In [12] we suggested a method which easily produces lots of stable triples, and yields much more informations about Kronecker coefficients. The idea is very general: we can study a space of sections of any line bundle L on any smooth irreducible variety X by taking the Taylor expansion of these sections along a smooth subvariety Y . To be more precise, we can define a filtration of $H^0(X, L)$ by the order of vanishing along Y ; more formally, we let

$$F_i = H^0(X, I_Y^i \otimes L) \subset H^0(X, L),$$

where I_Y is the ideal sheaf of Y . Let $\iota : Y \hookrightarrow X$ denote the inclusion map. The quotients $I_Y^i / I_Y^{i+1} = \iota_* S^i N^*$, for $N = \iota^* TX / TY$ the normal vector bundle of Y in X . Therefore there are natural injective maps

$$F_i / F_{i+1} \hookrightarrow H^0(Y, \iota^* L \otimes S^i N^*).$$

In words, this map takes a section of L vanishing to order i on Y , to the degree i part of its Taylor expansion in the directions normal to Y . The injectivity is clear: if the degree i part of the Taylor expansion is zero, then the section vanishes on Y at order $i + 1$.

In general, it will be quite difficult to determine the image of these maps. But the situation improves dramatically if we suppose that L is a sufficiently large tensor power of some given ample line bundle M on X . Indeed, it follows from the general properties of ample line bundles that for any fixed integer i , the map

$$F_i / F_{i+1} \hookrightarrow H^0(Y, \iota^* M^k \otimes S^i N^*)$$

must be surjective for k large enough. (This is a straightforward and very classical consequence of Serre's vanishing theorems for ample sheaves.)

Let us suppose moreover that the whole setting is preserved by the action of some reductive group G . Then our filtration splits as a filtration by G -submodules, and a splitting yields an injection of G -modules

$$H^0(X, M^k) \hookrightarrow H^0(Y, \iota^* M^k \otimes S^* N^*).$$

In fact this statement certainly holds true without any ampleness assumption: it simply asserts that an algebraic section of a line bundle is completely determined by its full Taylor expansion. In case M is ample, we have a very important extra information: we know that the right hand side is generated by the left hand side up to any given degree, if k is sufficiently large.

3.2 Application to Kronecker coefficients

We want to apply the previous ideas in the following situation. The variety X will be a flag manifold of a tensor product $V \otimes W$, not necessarily a complete flag; its type (by which we mean the sequence of dimensions of the subspaces in the flag) will be allowed to vary, so we will just denote it by $Fl_*(V \otimes W)$. The line bundle M will be some L_λ . If we need it to be ample the jumps in the partition λ will have to be given exactly by the type of the flag manifold, in which case we will say that λ is strict (one should add: relatively to the flag manifold under consideration). The subvariety Y of X will be a product $Fl_*(U) \times Fl_*(V)$ of flag manifolds of certain types, and we will need to construct the embedding ι . Of course our reductive group will be $G = GL(V) \times GL(W)$.

We have $Y = G/P$ for some parabolic subgroup P of G . It is well known that the category of G -equivariant vector bundles on Y is equivalent to the category of finite dimensional P -modules. Such modules can be quite complicated since P is not reductive. Let us consider a Levi decomposition $P = LU_P$, where U_P denotes the unipotent radical of P and L is a Levi factor, in particular a reductive subgroup of P . We can consider L -modules as P -modules with trivial action of U_P , and conversely. In general, any P -module has a canonical filtration by P -modules, such that the associated graded P -module has a trivial action of U_P . We can then consider it as an L -module and decompose it as a sum of irreducible modules, as in the usual theory of representations of reductive groups. Equivalently, any G -homogeneous vector bundle E as an associated graded homogeneous bundle $gr(E)$, which is a direct sum of irreducible ones. Cohomologically speaking, and even at the level of global sections, E and $gr(E)$ can be quite different. In fact E can in principle be reconstructed from $gr(E)$ through a series of extensions than can mix up the cohomology groups in a complicated fashion. Note however that $H^0(Y, E)$ is always a G -submodule of $H^0(Y, gr(E))$.

Example. On a complete flag variety, the irreducible bundles are exactly the line bundles. On a variety of incomplete flags $Fl_*(U)$ this is no longer true, but we can obtain them as follows. Denote by $p : Fl(U) \rightarrow Fl_*(U)$ the natural projection map (which takes a complete flag and simply forgets some of the subspaces in it). Consider a line bundle L_λ on $Fl(U)$. Then the pushforward $E_\lambda = p_* L_\lambda$ is an irreducible homogeneous vector bundle on $Fl_*(U)$ and they are all obtained in this way. (In particular, if λ is strict then E_λ is exactly the line bundle L_λ on $Fl_*(U)$, for which we use the same notation.) Note that the Borel-Weil theorem extends to all the irreducible bundles:

$$H^0(Fl_*(U), E_\lambda) = H^0(Fl(U), L_\lambda) = S_\lambda U.$$

Definition. The embedding ι is *stabilizing* if the normal bundle has the following convexity property: the highest weights of the positive degree part of the symmetric algebra $S^*(gr(N^*))$, considered as an L -module, are contained in some open half space of the weight lattice of L .

The line bundle $\iota^* L_\lambda$ will be of the form $L_{a(\lambda)} \otimes L_{b(\lambda)}$ for some weights $a(\lambda)$ and $b(\lambda)$ depending linearly on λ . More generally, given an irreducible homogeneous vector bundle E_α on $Fl_*(V \otimes W)$, we will need to understand its pull-back by ι . The resulting homogeneous bundle will scarcely be completely reducible. The associated graded bundle is of the form

$$gr(\iota^* E_\alpha) = \bigoplus_{(\rho, \sigma) \in T(\alpha)} E_\rho \otimes E_\sigma$$

for some multiset $T(\alpha)$.

Our main result is the following.

Theorem 2 *Suppose that the embedding ι is stabilizing. Let λ be any strict partition. Then*

1. The Kronecker coefficient $g(k\lambda, ka(\lambda), kb(\lambda)) = 1$ for any $k \geq 0$.
2. For any triple (α, β, γ) the Kronecker coefficient $g(\alpha + k\lambda, \beta + ka(\lambda), \gamma + kb(\lambda))$ is a non decreasing, bounded function of k , hence eventually constant. Otherwise said, the triple $(\lambda, a(\lambda), b(\lambda))$ is stable.
3. If moreover λ is ample, then the limit Kronecker coefficient is given by the multiplicity of the weight (β, γ) inside $gr(\iota^* E_\alpha) \otimes S^*(gr(N^*))$.
4. In particular, if α is strict, this is the multiplicity of $(\beta - a(\alpha), \gamma - b(\alpha))$ inside $S^*(gr(N^*))$.

Proof. The Kronecker coefficients $g(k\lambda, ka(\lambda), kb(\lambda))$ are positive since the restriction map

$$S_{k\lambda}(V \otimes W) = H^0(Fl_*(U), L_\lambda^k) \longrightarrow H^0(Fl_*(V) \times Fl_*(W), \iota^* L_\lambda^k) = S_{ka(\lambda)} V \otimes S_{kb(\lambda)} W$$

is surjective. Indeed this map is non zero since the line bundle L_λ is generated by global sections, which means that it has a section that does not vanish at any given point. The surjectivity then follows from Schur's lemma since the right hand side is irreducible. Moreover, we have seen that Taylor expansions induces a G -embedding

$$S_{k\lambda}(V \otimes W) \hookrightarrow H^0(Fl_*(V) \times Fl_*(W), \iota^* L_\lambda^k \otimes S^*(N^*)).$$

As we also noticed, replacing N^* by $gr(N^*)$ can only result in making the right hand side larger, so we even have an inclusion

$$S_{k\lambda}(V \otimes W) \hookrightarrow H^0(Fl_*(V) \times Fl_*(W), \iota^* L_\lambda^k \otimes S^*(gr(N^*))).$$

By the Borel-Weil theorem, the right hand side is now a sum of irreducible $GL(V) \times GL(W)$ -modules whose highest weights are of the form $(ka(\lambda), kb(\lambda))$ plus a highest weight of $S^*(gr(N^*))$. But since these weights are supposed to belong to a strictly convex cone, none can give a positive contribution of highest weight $(ka(\lambda), kb(\lambda))$ except the weight zero in degree zero. This proves that $g(k\lambda, ka(\lambda), kb(\lambda)) = 1$ for any $k \geq 0$.

Now consider an arbitrary triple (α, β, γ) . By the Borel-Weil theorem again, we have

$$S_{\alpha+k\lambda}(V \otimes W) = H^0(Fl_*(U), E_\alpha \otimes L_\lambda^k).$$

This immediately implies that $g(\alpha + k\lambda, \beta + ka(\lambda), \gamma + kb(\lambda))$ is a non decreasing function of k , by considering the map

$$H^0(Fl_*(U), E_\alpha \otimes L_\lambda^k) \otimes H^0(Fl_*(U), L_\lambda) \longrightarrow H^0(Fl_*(U), E_\alpha \otimes L_\lambda^{k+1})$$

and its restriction through ι . Moreover, the same approach using Taylor expansions can then be used without much change, in particular we get an injection

$$S_{\alpha+k\lambda}(V \otimes W) \hookrightarrow H^0(Fl_*(V) \times Fl_*(W), gr(\iota^* E_\alpha) \otimes \iota^* L_\lambda^k \otimes S^*(gr(N^*))).$$

The same argument as before then implies that $g(\alpha + k\lambda, \beta + ka(\lambda))$ is bounded by the multiplicity of the representation of highest weight (β, γ) inside $gr(\iota^* E_\alpha) \otimes S^*(gr(N^*))$, which is finite by the convexity hypothesis.

What remains to check is that for k large enough, we have in fact equality. But being finite, the multiplicity (β, γ) inside $gr(\iota^* E_\alpha) \otimes S^*(gr(N^*))$ only comes from some finite part of the symmetric algebra of the conormal bundle. Then it follows formally from the properties of ample line bundles that if k is large enough, all the maps involved in the Taylor expansions are surjective in bounded degrees, and it makes no difference to replace all the homogeneous bundles involved by the associated graded bundles. This concludes the proof. \square

Corollary 2 *The set of triples $(\lambda, a(\lambda), b(\lambda))$ is a face of the Kronecker polyhedron. Moreover the local structure of the polyhedron around this face is given by the cone generated by the weights of $S^*(gr(N^*))$.*

Corollary 3 *Limit Kronecker coefficients have the monotonicity property.*

In particular, if they are non zero, then they remain nonzero after stretching. Kirillov [9] and Klyachko [10] conjectured in certain special cases that the converse property, that is saturation, should hold. We could ask more generally if our limit (or reduced) Kronecker coefficients should have the saturation property, but we actually have only very limited evidence for that.

Remarks.

1. We have not tried to bound the minimal value of k starting from which the Kronecker coefficient $g(\alpha + k\lambda, \beta + ka(\lambda), \gamma + kb(\lambda))$ does stabilize, but it would in principle be possible to give an effective version of the previous theorem. Indeed an irreducible homogeneous bundle which has non zero sections has no higher cohomology by Bott's theorem, and this would be sufficient to ensure for example that replacing the conormal bundle by its graded associated bundle makes no difference at the level of global sections. Nevertheless the resulting statements would probably be rather heavy, and presumably not even close to be sharp.
2. We have expressed the limit multiplicities in a rather compact form, but we will see that these expressions are in fact very complicated. Moreover they usually involve other Kronecker coefficients, as well as Littlewood-Richardson coefficients and other interesting variants. It seems we are playing with Russian dolls, but with growing complexity when we open a doll to find a smaller, more mysterious one...

3.3 Tangent and normal bundles

In order to apply the previous theorem, we need to be able to understand the normal bundle of our embedding ι and the associated graded bundle. The starting point for that is of course to understand the tangent bundle to a flag variety; this is classical and goes as follows.

Suppose we consider a variety $Fl_*(V)$ of flags $(0 = V_0 \subset V_1 \subset \dots \subset V_{r-1} \subset V_r = V)$ where V_i has dimension d_i . Each of these spaces defines a homogeneous vector bundle on $Fl_*(V)$, but not irreducible in general. The irreducible homogeneous bundles are obtained by considering the quotient bundles $Q_i = V_i/V_{i-1}$, for $1 \leq i \leq r$. Each of these bundles is irreducible, as well as each of their Schur powers. More generally, each irreducible vector bundle on $Fl_*(V)$ is of the form

$$E_\lambda = S_{\alpha_1} Q_1 \otimes \dots \otimes S_{\alpha_r} Q_r$$

for some non increasing sequences $(\alpha_1, \dots, \alpha_r)$ of relative integers. Each homogeneous vector bundle F can then be constructed from such irreducible bundles by means of suitable extensions; the set of irreducible bundles involved does not depends on the process and their sum is $gr(F)$. For example we have non trivial exact sequences $0 \rightarrow V_{i-1} \rightarrow V_i \rightarrow Q_i \rightarrow 0$, and by induction we deduce that

$$gr(V_i) = Q_1 \oplus \dots \oplus Q_i.$$

The tangent bundle to $Fl_*(V)$ at the flag $V_\bullet = (0 = V_0 \subset V_1 \subset \dots \subset V_{r-1} \subset V_r = V)$ can be naturally identified with the quotient of $End(V)$ by the subspace of endomorphisms preserving V_\bullet . Taking the orthogonal with respect to the Killing form we get a natural identification of the cotangent bundle:

$$T_{Fl_*(V)}^* \simeq \{X \in End(V), X(V_i) \subset V_{i-1}, 1 \leq i \leq r\}.$$

From this we can easily deduce the following statement:

Lemma 1 *The associated bundle of the tangent bundle of $Fl_*(V)$ is*

$$gr(T_{Fl_*(V)}) = \oplus_{1 \leq i < j \leq r} Hom(Q_i, Q_j).$$

In order to describe the normal bundle of an embedding $\iota : Fl_*(V) \times Fl_*(W) \hookrightarrow FL_*(V \otimes W)$, we will have to take the quotients of two such bundles and it is clear that the result will be quite complicated in general. Rather than writing down general formulas, we will compute these quotients in certain specific situations; mainly, when there are only few terms (typically for Grassmannians, whose tangent bundles are irreducible) or for complete flag varieties (whose irreducible bundles are line bundles, hence much easier to handle).

4 First examples

We now have all the necessary ingredients in hand in order to apply Theorem 2. What remains to be done is to construct suitable embeddings between flag varieties and determine whether they are stabilizing. We begin with a few simple examples.

4.1 Murnaghan's stability

As an appetizer we begin with a well-known instance of stability, first discovered by Murnaghan [19, 20]. This is the statement that for any triple of partitions (α, β, γ) , the Kronecker coefficient $g(\alpha + (k), \beta + (k), \gamma + (k))$ is eventually constant for large k (a sharp bound on k has been given in [3]). The limit value is called a reduced Kronecker coefficient and is denoted $\bar{g}(\alpha', \beta', \gamma')$, where α' is deduced from α by suppressing the first part.

To interpret this in our setting, consider the Segre embedding

$$\iota : \mathbf{P}(V) \times \mathbf{P}(W) \hookrightarrow \mathbf{P}(V \otimes W).$$

If L denotes the tautological line bundle on projective space, then with some abuse of notations, $\iota^*L = L \otimes L$. From Lemma 1 we easily derive that

$$N = Hom(L, V/L) \otimes Hom(L, W/L).$$

In particular this embedding is clearly stabilizing since any component of S^*N will have negative degree on both copies of L . This implies Murnaghan's stability right away.

Moreover, if Q is the quotient bundle on $\mathbf{P}(V \otimes W)$, then the associated bundle of ι^*Q is

$$gr(\iota^*Q) = L \otimes W/L \oplus V/L \otimes L \oplus V/L \otimes W/L.$$

Applying Theorem 2, we deduce that $\bar{g}(\alpha', \beta', \gamma')$ is the multiplicity of $S_{\beta'}A \otimes S_{\gamma'}B$ inside

$$S_{\alpha'}(A \otimes B \oplus A \oplus B) \otimes S^*(A \otimes B).$$

This is equivalent to Lemma 2.1 in [3]. The monotonicity property that is a special case of Corollary 3 has been observed in [5].

4.2 k -stability

A generalisation of Murnaghan's stability has been considered recently by Vallejo [26] and Pak and Panova [21]. This k -stability, as coined by the latter authors, corresponds to the generalized Segre embedding

$$\iota : \mathbf{P}(V) \times Gr(k, W) \hookrightarrow Gr(k, V \otimes W),$$

sending a pair (L, M) made of a one dimensional subspace $L \subset V$ and a k dimensional subspace $M \subset W$, to the k dimensional subspace $L \otimes M$ of $V \otimes W$. If L denotes the tautological line bundle on the Grassmann variety, then $\iota^*L = L^k \otimes L$. Extending the computation we made for projective spaces we get that

$$grN = Hom(L, V/L) \otimes End_0(M) \oplus Hom(L, V/L) \otimes Hom(M, W/M).$$

For essentially the same reasons as in the previous case, this embedding is manifestly stabilizing. This implies Theorem 1.1 in [21] without further ado. Theorem 10.2 in [26] gives an effective version.

Moreover we have access to the limit multiplicities, called the k -reduced Kronecker coefficients. For this we need to consider a vector bundle E_λ on $Gr(k, V \otimes W)$ and pull it back by ι . Since E_λ is a Schur power of a the quotient bundle Q and

$$gr(\iota^*Q) = L \otimes W/M \oplus V/L \otimes W \oplus V/L \otimes W/M,$$

we get the following result.

Proposition 3 *The k -reduced Kronecker coefficient $\bar{g}_k(\lambda, \mu, \nu)$ is equal to the multiplicity of the product $A_-^{\mu_-} \otimes S_{\mu_+} A_+ \otimes S_{\nu_-} B_- \otimes S_{\nu_+} B_+$ inside*

$$S_\lambda(A_- \otimes B_+ \oplus A_+ \otimes B_- \oplus A_+ \otimes B_+) \otimes Sym(A_-^* \otimes A_+ \otimes End_0(B_-)) \otimes Sym(A_-^* \otimes A_+ \otimes B_-^* \otimes B_+).$$

For Geometric Complexity Theory the most relevant Kronecker coefficients are those indexed by two partitions with equal rectangular shapes [4]. This corresponds to taking λ and ν equal to the empty partition. To get a contribution from the previous formula we must avoid all the terms contributing positively to B_+ , which kills the second symmetric algebra. The first will contribute through $S_\mu A_+ \otimes S_\mu(End_0(B_-))$ (because of the Cauchy formula), and extracting the term with $\nu_- = 0$ means that we take the $GL(B_-)$ -invariants inside $S_\mu(End_0(B_-))$. Since $End_0(B_-)$ is just sl_k , we get:

Corollary 4 *The k -reduced Kronecker coefficient $\bar{g}_k(0, \mu, 0)$ is equal to the dimension of the GL_k -invariant subspace of $S_\mu(sl_k)$.*

An effective version of this result was derived in [15] using a completely different method.

It seems interesting to notice that Schur powers of $sl(V) = End_0(V)$ are often involved in the stable Kronecker coefficients, along with the Kronecker coefficients themselves and the Littlewood-Richardson coefficients. These multiplicities are also of interest for themselves; more generally, many interesting phenomena appear when one considers Schur powers of the adjoint representation of a simple complex Lie algebra, see eg [11]. Our methods can be applied to the study of the asymptotics of these coefficients; we hope to come back to this question in a future paper.

4.3 Grassmannian stability

We can obviously generalize the Segre embedding to any product of Grassmann varieties. For any positive integers a, b consider the natural embedding

$$\iota : Gr(a, V) \times Gr(b, W) \hookrightarrow Gr(ab, V \otimes W).$$

If L denotes the tautological line bundle on the Grassmann variety, then $\iota^*L = L^b \otimes L^a$. Extending the computation we made for projective spaces we get that

$$grN = End_0(A) \otimes Hom(B, W/B) \oplus Hom(A, V/A) \otimes End_0(B) \oplus Hom(A, V/A) \otimes Hom(B, W/B).$$

This embedding is again stabilizing and we get right away the following generalization of k -stability.

Proposition 4 *Let a, b be positive integers. For any triple (λ, μ, ν) , the Kronecker coefficient*

$$g(\lambda + (t)^{ab}, \mu + (at)^b, \nu + (bt)^a)$$

is a non decreasing, eventually constant, function of t .

We could call the limits (a, b) -reduced Kronecker coefficients, and provide an expression generalizing Theorem 3. We leave this to the interested reader. Note that in this statement (λ, μ, ν) are not just partitions but integer sequences for which the arguments of the Kronecker coefficient are partitions, at least for large enough t . For example $\lambda = (\lambda_+, \lambda_-)$ where λ_+ is a non decreasing sequence of length ab (with possibly negative entries), and λ_- is a partition (of arbitrary length).

Note that when λ_- is the empty partition, drastic simplifications occur. This follows from the usual formula $S_{\alpha+(t)^d} V = S_\alpha V \otimes (\det V)^t$ when d is the dimension of V . Since $\det(A \otimes B) = (\det A)^{\dim B} \otimes (\det B)^{\dim A}$, we deduce that

$$g(\lambda + (t)^{ab}, \mu + (at)^b, \nu + (bt)^a) = g(\lambda, \mu, \nu).$$

This is Theorem 3.1 in [25].

5 Stability and standard tableaux

5.1 Classification of embeddings

In this section we focus on the equivariant embeddings of complete flag varieties

$$\iota : Fl(A) \times Fl(B) \hookrightarrow Fl(A \otimes B),$$

the combinatorics being more transparent in that case. Denote by a and b the respective dimensions of A and B . The following two statements appear in [12]:

Proposition 5 *The equivariant embeddings $\iota : Fl(A) \times Fl(B) \hookrightarrow Fl(A \otimes B)$ are classified by standard tableaux of rectangular shape $a \times b$.*

Let T be such a standard tableau of rectangular shape $a \times b$. The corresponding embedding ι_T is defined as follows. Let A_\bullet and B_\bullet be two complete flags in A and B respectively. Choose adapted basis u_1, \dots, u_a of A and v_1, \dots, v_b of B , that is, such that $A_i = \langle u_1, \dots, u_i \rangle$ and $B_j = \langle v_1, \dots, v_j \rangle$. Then define the complete flag C_\bullet in $A \otimes B$ by

$$C_k = \langle u_i \otimes v_j, T(i, j) \leq k \rangle.$$

Here $T(i, j)$ denotes the entry of T in the box (i, j) . Clearly this flag does not depend on the adapted basis but only on the flags, and we can let $\iota_T(A_\bullet, B_\bullet) = C_\bullet$.

Proposition 6 *The embedding $\iota_T : Fl(A) \times Fl(B) \hookrightarrow Fl(A \otimes B)$ is stabilizing if and only if the standard tableau T is additive.*

Here we use the terminology of Vallejo, the additivity condition being used in [27]. In our paper [12] no special terminology was introduced for this additivity property, which means the following: there exist increasing sequences $x_1 < \dots < x_a$ and $y_1 < \dots < y_b$ such that

$$T(i, j) < T(k, l) \iff x_i + y_j < x_k + y_l.$$

Remark. There is a huge number of embeddings ι_T : recall that by the hook length formula, the number of standard tableaux of shape $a \times b$ is $ST(a, b) = (ab)!/h(a, b)$, where $h(a, b) = (a + b - 1)!!/(a - 1)!!(b - 1)!!$. This grows at least exponentially with a and b . Among these, the proportion of additive tableaux probably tends to zero when a and b grow, but their number should still grow exponentially. Note that each of this additive tableau corresponds to a certain chamber in the complement of the arrangement of hyperplanes H_{ijkl} defined by the equalities $x_i + y_j = x_k + y_l$ in \mathbf{R}_+^{a+b-2} (we may suppose that $x_1 = y_1 = 0$). This looks very much like the hyperplane arrangement associated to a root system.

The partitions $a_T(\lambda)$ and $b_T(\lambda)$ such that $\iota_T^* L_\lambda = L_{a_T(\lambda)} \otimes L_{b_T(\lambda)}$ are easily described; one just needs to read the entries in each line or column of T and sum the corresponding parts of λ :

$$a_T(\lambda)_i = \sum_{j=1}^b \lambda_{T(i,j)}, \quad b_T(\lambda)_j = \sum_{i=1}^a \lambda_{T(i,j)}.$$

5.2 (T, λ) -reduced Kronecker coefficients

Applying our general statements of Theorem 2, we get:

Proposition 7 *Let T be any additive tableau. For any partition λ , the triple $(\lambda, a_T(\lambda), b_T(\lambda))$ is stable.*

Although this statement does not appear explicitly in [12], it is discussed p.735, in the paragraph just before the Example. It was recently rediscovered by Vallejo [27], inspired by the work of Stembridge [24], and following a completely different approach. (As a matter of fact there is a slight difference between our definition of additivity and that of Vallejo. When the tableau T is additive, as we intend it, then for any partition λ of length at most ab , the matrix A with entries $a_{ij} = \lambda_{T(i,j)}$ is additive in the sense of Vallejo, and all such matrices are obtained that way. As we just stated it, this implies that the triple $(\lambda, a_T(\lambda), b_T(\lambda))$ is stable. If moreover λ is strict, then we will get more information, in particular we will be able to compute the stable Kronecker coefficients.)

Note that the additivity property is introduced in section 3.1.2 of [12] and explained to be equivalent to the convexity property of the normal bundle that implies stability. At that time we were mainly interested in plethysm and we treated the case of Kronecker coefficients rather quickly, giving details only for a sample of the results that were amplified in the strongly similar case of plethysm. Also we treated directly the multiKronecker coefficients

$$g(\mu_1, \dots, \mu_r) = \dim([\mu_1] \otimes \dots \otimes [\mu_r])^{S_n},$$

for which the method applies with essentially no difference.

The fact that additivity is equivalent to stability is easy to understand. Recall, as a special case of Lemma 1, that the tangent bundle of a complete flag manifold $Fl(U)$ has associated graded bundle

$$gr(T_{Fl(U)}) = \oplus_{i < j} Hom(Q_i, Q_j),$$

where the quotient bundles are now line bundles. Applying this to $U = V \otimes W$ and pulling-back by ι_T we will get a formula in terms of the quotient line bundles on $Fl(V)$ and $Fl(W)$, that we will denote by E_i and F_j . The formula reads

$$gr(\iota_T^* T_{Fl(V \otimes W)}) = \oplus_{T(i,j) < T(k,l)} Hom(E_i \otimes F_j, E_k \otimes F_l).$$

Let us denote by e_1, \dots, e_a and f_1, \dots, f_b natural basis of the weight lattices of $GL(V)$ and $GL(W)$, respectively. The formula shows that the weights of the restricted tangent bundle

$\iota_T^*(T_{Fl(V \otimes W)})$ are the $e_k + f_l - e_i - f_j$ for $T(i, j) < T(k, l)$. We claim that the weights of the normal bundle are exactly the same. Indeed, since the tableau T is increasing on rows and columns, the weights $e_k - e_i$ and $f_l - f_j$ appear for $k > i$ and $l > j$ with multiplicity b and a respectively, in particular greater than one. Since they are the weights of the tangent bundles of $Fl(V)$ and $Fl(W)$, we see that when we go from the restricted tangent bundle to the normal bundle the list of weights will not change, only certain multiplicities will decrease, but remaining positive. In particular the generated cone will not be affected.

Finally, recall that the additivity property asks for the existence of sequences $x_1 < \dots < x_a$ and $y_1 < \dots < y_b$ such that $T(i, j) < T(k, l)$ if and only if $x_i + y_j < x_k + y_l$, otherwise written as $x_k - x_i + y_l - y_j > 0$. This precisely means that the corresponding linear form is positive on each of the weights $e_k - e_i + f_l - f_j$ with $T(i, j) < T(k, l)$.

Remark. Of course it is not necessary, in order to determine the cone generated by the weights of $gr(N^*)$, to compile the full list of $M = (a-1)(b-1)(ab+a+b)/2$ weights. The $ab-1$ of them obtained by reading the successive entries of the tableau T from 1 to ab will suffice, since all the other weights will obviously be sums of these.

Note that from Proposition 7, we can deduce right away the following special case, which is a generalization of Proposition 4.

Corollary 5 *For any partition μ of size m , the triple $(1^m, \mu, \mu^*)$ is stable.*

Proof. Consider a rectangle $a \times b$ in which the diagram of μ can be inscribed. Consider μ and the conjugate partition μ^* as integer sequences of lengths a and b respectively, by adding zeroes if necessary. Consider the increasing sequences $\alpha_1, \dots, \alpha_a$ and β_1, \dots, β_b defined by $\alpha_i = i - \mu_i - 1$ and $\beta_j = j - \mu_j^*$. Then $\alpha_i + \beta_j = -h_{ij}$, the opposite of the hook length of μ for the box (i, j) . In particular $\alpha_i + \beta_j$ is negative exactly on the support of μ . Let T be the corresponding additive tableau, and let $\lambda = 1^m$. Then $a_T(\lambda) = \mu$ and $b_T(\lambda) = \mu^*$. Hence $(1^m, \mu, \mu^*)$ is stable. \square

Let us denote the value of $g(\alpha + k\lambda, \beta + ka_T(\lambda), \gamma + kb_T(\lambda))$, for k very large, by $g_{T,\lambda}(\alpha, \beta, \gamma)$, and call it a (T, λ) -reduced Kronecker coefficient. If λ is strictly decreasing of length ab , or $ab-1$, so that the corresponding line bundle on the flag manifold is very ample, we have seen that this (T, λ) -reduced Kronecker coefficient can be computed as the multiplicity of the weight $(\beta - a_T(\alpha), \gamma - b_T(\alpha))$ inside the symmetric algebra $S^*(gr(N^*))$. The weights of $gr(N^*)$ can readily be read off the tableau T . Let us denote them by (u_i, v_i) , for $1 \leq i \leq M = (a-1)(b-1)(ab+a+b)/2$. Denote by $P_{T,\lambda}(\mu, \nu)$ the polytope defined as the intersection of the quadrant $t_1, \dots, t_M \geq 0$ in \mathbf{R}^M with the affine linear space defined by the condition that

$$\sum_{i=1}^M t_i(u_i, v_i) = (\mu, \nu).$$

Proposition 8 *The (T, λ) -reduced Kronecker coefficient $g_{T,\lambda}(\alpha, \beta, \gamma)$ is equal to the number of integral points in the polytope $P_{T,\lambda}(\beta - a_T(\alpha), \gamma - b_T(\alpha))$.*

Of course we would then be tempted to stretch the triple (α, β, γ) . We would then obtain the stretched (T, λ) -reduced Kronecker coefficient $g_{T,\lambda}(k\alpha, k\beta, k\gamma)$ as given by the Ehrhart quasi-polynomial of the rational polytope $P_{T,\lambda}(\beta - a_T(\alpha), \gamma - b_T(\alpha))$. This suggests interesting behaviours for multistretched Kronecker coefficients, but we will not pursue on this route.

Let us simply notice an obvious consequence for (T, λ) -reduced Kronecker coefficients, the following translation invariance property:

$$g_{T,\lambda}(\alpha, \beta, \gamma) = g_{T,\lambda}(\alpha + \delta, \beta + a_T(\delta), \gamma + b_T(\delta))$$

for any partition δ .

5.3 Faces of the Kronecker polytope

As we observed, the convexity property of the embeddings defined by additive tableaux has very interesting consequences for the Kronecker polytope. Let us first reformulate Corollary 2.

Proposition 9 *Each additive tableau T defines a regular face f_T of the Kronecker polyhedron $PKron_{a,b,ab}$, of minimal dimension.*

Regular means that the face meets the interior of the Weyl chamber, which is the set of strictly decreasing partitions. Ressayre proved in [23], in a more general setting, that the maximal codimension of a regular face is the rank of the group which in our case is $GL(V) \times GL(W)$. This exactly matches with the codimension of f_T .

Around this minimal face f_T , we know that the local structure of $PKron_{a,b,ab}$ is described by the convex polyhedron generated by the weights of $gr(N^*)$.

A face of our polytope will be defined by a linear function which is non negative on all these vectors, and vanishes on a subset of them that generate a hyperplane. Such a linear function will be defined by sequences $x_1 \leq \dots \leq x_a$ and $y_1 \leq \dots \leq y_b$ such that $x_i + y_j \leq x_k + y_l$ when $T(i, j) < T(k, l)$. The different values of $x_i + y_j$ define a partition of the rectangle $a \times b$ into disjoint regions, and this partition is a *relaxation* of T , in the sense that each region is numbered by consecutive values of T . The hyperplane condition can be interpreted as the fact that the vectors $e_k - e_i + f_l - f_j$, for (i, j) and (k, l) belonging to the same region, generate a hyperplane in the weight space. This is also a maximality condition: we cannot relax any further while keeping the compatibility condition with T . We deduce:

Proposition 10 *The facets F_R of the Kronecker polyhedral cone $Kron_{a,b,ab}$ containing the minimal face f_T are in bijective correspondence with the maximal relaxations R of the tableau T .*

Example. Recall that for $a = b = 3$ there are 42 standard tableaux fitting in a square of size three, among which 36 are additive. The total number of maximal relaxations of these additive tableaux is 17. Up to diagonal symmetry they are as follows, where each square is filled by the entry $x_i + y_j$ in box (i, j) for some sequences $(x_1 = 0, x_2, x_3)$ and $(y_1 = 0, y_2, y_3)$.

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 2 & 3 & 3 & 3 & 4 & 5 \\ \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & 1 & 2 & 2 & 2 & 3 & 1 & 2 & 2 & 2 & 3 & 4 & 3 & 4 & 5 \end{array}$$

Consider for example the relaxation R encoded in the tableau

$$\begin{array}{ccc} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{array}$$

It splits the square into six regions, three of size one and three of size two. There are therefore eight compatible standard tableaux T , which are all additive. This gives eight minimal faces f_T incident to the facet F_R .

Proposition 11 *The defining inequalities of the facet F_R associated to a maximal relaxation R defined by sequences (x_i, y_j) are of the form*

$$\sum_{i=1}^a x_i \beta_i + \sum_{j=1}^b y_j \gamma_j \geq \sum_{i=1}^a \sum_{j=1}^b (x_i + y_j) \alpha_{T(i,j)},$$

where T is any standard tableau compatible with R .

We mean that R is a relaxation of T . It is clear then that the inequality does not depend on T , since taking another compatible tableau T' amounts to switching some entries only inside the regions defined by R , on each of which the sum $x_i + y_j$ is constant.

5.4 From rectangles to arbitrary tableaux

If we want to restrict to the Kronecker cone $Kron_{a,b,c}$ for some $c < ab$, we just need to intersect $Kron_{a,b,ab}$ with a linear space L_c of codimension $ab - c$, which meets each of our minimal faces f_T . But this raises two issues. First, two distinct minimal faces can have the same intersection with L_c . Second, it is not clear whether the intersection of a facet F_R of $Kron_{a,b,ab}$ with L_c will still be a facet of $Kron_{a,b,c}$.

The first issue is easy to address: two tableaux T and T' will give the same minimal face of $Kron_{a,b,c}$ if and only if they coincide up to c , that is, their entries smaller or equal to c appear in the same boxes. We thus get minimal faces of $Kron_{a,b,c}$ parametrized by standard tableaux S of size c inside the rectangle $a \times b$, which are additive in the same sense as before.

To address the second issue we can modify our embeddings ι_T accordingly. A standard tableau S of size c inside the rectangle $a \times b$ defines an embedding

$$\iota_S : FL(V) \times Fl(W) \hookrightarrow FL_c(V \otimes W),$$

where we denote by $FL_c(U)$ the partial flag manifold of U parametrizing flags of the form $(0 = U_0 \subset U_1 \subset \dots \subset U_c \subset U)$, where U_i has dimension i . We will require that S fits exactly in the rectangle, not in any smaller one. Denote the quotient bundles on $Fl(U)$ by Q_1, \dots, Q_c, Q_{c+1} , they are all line bundles except the last one. We have $\iota_S^* Q_k = E_i \otimes F_j$ whenever $S(i, j) = k \leq c$, while $\iota_S^* Q_{c+1}$ is not irreducible, but has associated graded bundle

$$gr(\iota_S^* Q_{c+1}) = \oplus_{(i,j) \notin S} E_i \otimes F_j.$$

The pull-back of the tangent bundle is then given by the same formula as before,

$$gr(\iota_S^* T_{Fl(V \otimes W)}) = \oplus_{S(i,j) < S(k,l)} Hom(E_i \otimes F_j, E_k \otimes F_l),$$

except that for this to be correct, we need to consider S as a rectangular tableau of size $a \times b$, in which the boxes (i, j) that do not belong to S are all numbered by the same arbitrarily large number, say $S(i, j) = \infty$. Exactly as before we deduce that the weights of the normal bundle are the $e_k - e_i + f_l - f_j$ with $S(i, j) < S(k, l)$, and the convexity condition translates into the same additive property, that we can summarize by saying that S must be a piece of a rectangular additive standard tableau. The discussion above then goes through exactly as in the rectangular case, except that we don't need to care about the boxes of the rectangle that are not supported by S . We get the following slight extension of our previous results:

Proposition 12 *Let S be a standard tableau of height a , width b , size c . Suppose that S is additive. Then the set of stable triples of the form $(\lambda, a_S(\lambda), b_S(\lambda))$ defines a minimal regular face f_S of the Kronecker polytope $PKron_{a,b,c}$. Moreover the facets of the polytope containing this minimal face f_S are in bijection with the minimal relaxations R of S . If R is defined by non decreasing sequences (x_i, y_j) , the equation of the facet F_R is*

$$\sum_{i=1}^a x_i \beta_i + \sum_{j=1}^b y_j \gamma_j \geq \sum_{(i,j) \in S} (x_i + y_j) \alpha_{S(i,j)}.$$

Remark. More variants could be explored. Maps $Fl_*(V) \times Fl_*(W) \hookrightarrow Fl_*(V \otimes W)$ where arbitrary types appear at the source are easily constructed in terms of tableaux, and their stability could be analyzed. This will be more complicated in general since the quotient bundles will have ranks bigger than one. Moreover it will only give access to stable triples on the boundary of the Weyl chamber.

Of course we could also readily extend the discussion to an arbitrary number of vector spaces, describe embeddings of arbitrary products of flag manifolds in terms of multidimensional tableaux, observe that the convexity condition on the weights of the normal bundle is again an additivity condition, and deduce stability properties for multiKronecker coefficients. This was partly done in [12].

6 Rectangles: stability and beyond

There exist stable triples which do not come from additive tableaux, and it would be nice to understand them. Stembridge in [24] observed that $(22, 22, 22)$ is stable, and it is certainly not additive. (Nevertheless it is highly degenerate, in the sense that it belongs to a very small face of the dominant Weyl chamber. In particular this example leaves open the question of the existence of non additive stable triples in the interior of the Weyl chamber.) In this section we make a connection with Cayley's hyperdeterminant and the Dynkin diagram D_4 , and we explain another observation by Stembridge in terms of affine E_6 .

6.1 Finite cases

The rectangular Kronecker coefficients, ie those involving partitions of rectangular shape, are of special interest because of their direct relation with invariant theory. For three factors,

$$g(p^a, q^b, r^c) = \dim(S^k(A \otimes B \otimes C))^{SL(A) \times SL(B) \times SL(C)}$$

when $k = pa = qb = rc$ and a, b, c are the dimensions of A, B, C . Of course this connection also holds for a larger number of factors.

There seems to exist only few results on these quotients. The cases for which there are only finitely many orbits of $GL(A) \times GL(B) \times GL(C)$ inside $A \otimes B \otimes C$ have been completely classified in connection with Dynkin diagrams (see eg [16] and references therein). One can deduce all the possible dimensions a, b, c for which there exists a dense orbit, through the combinatorial process called casting transforms. In this situation there exists one invariant for $SL(A) \times SL(B) \times SL(C)$ at most, depending on the codimension of the complement of the dense open orbit, which can be one (in which case its equation is an invariant) or greater than one (in which case there is no invariant). The former case gives a weakly stable triple.

In this classification through Dynkin diagrams, the triple tensor products correspond to triple nodes, so there are only few cases, coming from diagrams of type D or E. The first interesting case is D_4 , corresponding to $a = b = c = 2$. The invariant is the famous hyperdeterminant first discovered by Cayley, which has degree four. This implies that $g(n^2, n^2, n^2) = 1$ when n is even and $g(n^2, n^2, n^2) = 0$ when n is odd. In particular, $(22, 22, 22)$ is a weakly stable triple, and even a stable triple, as shown by Stembridge [24].

The next two diagrams, D_5 and D_6 , give $(a, b, c) = (2, 2, 3)$ and $(2, 2, 4)$ respectively. The corresponding invariants have degree 6 and 4. They correspond to the weakly stable triples $(33, 33, 222)$ and $(22, 22, 1111)$ (which we already met among stable triples). For D_n , $n \geq 7$, we get $(a, b, c) = (2, 2, n - 2)$ but there is no non trivial invariant anymore. Finally the triple nodes of E_6, E_7, E_8 yield the triples $(a, b, c) = (2, 3, 3), (2, 3, 4), (2, 3, 5)$. There is no invariant for the latter case, but an invariant of degree 12 in the two previous ones, yielding the weakly stable triples $(66, 444, 444)$ and $(66, 444, 3333)$. Let us summarize our discussion:

Proposition 13 *The triple nodes of the Dynkin diagrams of types D_4 , D_5 , E_6 , E_7 yield the non additive weakly stable triples $(22, 22, 22)$, $(33, 33, 222)$, $(66, 444, 444)$, $(66, 444, 3333)$.*

6.2 Affine cases

This discussion can be upgraded from Dynkin to affine Dynkin diagrams. Indeed it is a theorem of Kac [8] that when we consider a representation associated to a node of an affine Dynkin diagram, the invariant algebra is free, or otherwise said, is a polynomial algebra. (If the chosen node is the one that has been attached to the usual Dynkin diagram, the associated representation is just the adjoint one, so the theorem generalizes the well-known result that the invariant algebra of the adjoint representation is free.)

There will be four cases related to Kronecker coefficients, corresponding to the affine Dynkin diagrams with a unique multiple node. Let us introduce the following notation:

$$\begin{aligned} g_{\hat{D}_4}(n) &= g(n^2, n^2, n^2, n^2), \\ g_{\hat{E}_6}(n) &= g(n^3, n^3, n^3), \\ g_{\hat{E}_7}(n) &= g((2n)^2, n^4, n^4), \\ g_{\hat{E}_8}(n) &= g((3n)^2, (2n)^3, n^6). \end{aligned}$$

Applying Kac's results we immediately identify the generating series of these rectangular Kronecker coefficients.

Proposition 14 *The generating series of the rectangular Kronecker coefficients $g_{\hat{D}_4}(n)$, $g_{\hat{E}_6}(n)$, $g_{\hat{E}_7}(n)$ and $g_{\hat{E}_8}(n)$ are the following:*

$$\begin{aligned} \sum_{n \geq 0} g_{\hat{D}_4}(n) q^n &= \frac{1}{(1-q)(1-q^2)^2(1-q^3)}, \\ \sum_{n \geq 0} g_{\hat{E}_6}(n) q^n &= \frac{1}{(1-q^2)(1-q^3)(1-q^4)}, \\ \sum_{n \geq 0} g_{\hat{E}_7}(n) q^n &= \frac{1}{(1-q^2)(1-q^3)}, \\ \sum_{n \geq 0} g_{\hat{E}_8}(n) q^n &= \frac{1}{1-q}. \end{aligned}$$

The last of these identities simply expresses the fact that $(33, 222, 1^6)$ is a weakly stable triple, as we already know. The previous one can be rewritten as

$$g((2n)^2, n^4, n^4) = \frac{n + \pi_6(n)}{6}$$

where π_6 is the 6-periodic function with first 6 values $(6, -1, 4, 3, 2, 1)$. The identity for $g_{\hat{E}_6}(n)$ has been suggested by Stembridge ([24], Appendix). It can also be rewritten as a quasipolynomial identity:

$$g(n^3, n^3, n^3) = \frac{(n+1)(n+2)}{48} + \frac{n+1}{16}\pi_2(n) + \frac{1}{48}\pi_{12}(n),$$

where π_{12} is 12-periodic with period $(37, -12, 9, 16, 21, -48, 25, 0, 21, 4, 9, 0)$ and π_2 is 2-periodic with period $(3, 1)$. Finally the quadruple Kronecker coefficient

$$g(n^2, n^2, n^2, n^2) = \frac{n^3 + 12n^2 + 29n + 18}{72} + \frac{n+1}{72}\pi_2(n) + \frac{1}{72}\pi_6(n),$$

where π_6 is 6-periodic with period $(35, -8, 27, 8, 19, 0)$ and π_2 is 2-periodic with period $(19, 10)$.

Remark. Let us mention that there should exist lots of non additive stable triples. For example, [15] implies that if λ is a partition of size $2n$, then (n^2, n^2, λ) is weakly stable when λ is even and of length at most four, as well as $(n^4, (2n)^2, 2\lambda)$ when λ has length at most three and $\lambda_1 \leq \lambda_2 + \lambda_3$. It would be interesting to prove that these weakly stable triples are in fact stable, and to get more examples, or more general procedures to construct (weakly) stable triples.

Theorem 6.1 in [24] gives a criterion for stability that covers the additive triples, but not only those. It would be interesting to understand these non additive triples more explicitly, find a geometric interpretation, compute the stable Kronecker coefficients, and decide to which extent they could help to describe the Kronecker polyhedra.

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